MTH 403: Real Analysis II

Practice Assignment II

Conceptual problems

1. Try Problem 2-36 (page 29) from *Calculus on manifolds* by M. Spivak. (This problem is essential for the theorem on the existence of partition of unity.)

2. A bounded set $C \in \mathbb{R}^n$ is said to be *Jordan measurable* if ∂C has measure 0. Show that if a set A if Jordan measurable, then given $\epsilon > 0$, there exists a compact Jordan measurable set $C \subset A$ such that $\int_{A \setminus C} 1 < \epsilon$. (Note that $\int_C 1$ is called the *content* or the *volume* of the set C.)

3. Let $C \subset V \subset \mathbb{R}^n$, where C is a compact set and V is an open set. Show that there exists a compact set D such that $C \subset D^\circ \subset D \subset V$.

4. Try Problem 3-35 (page 62) from *Calculus on manifolds* by M. Spivak.

5. Try Problem 3-41 (page 73) from *Calculus on manifolds* by M. Spivak.

Integration

We will use Q to denote a rectangle in $\mathbb{R}^n$ and $\mathbb{Q}$ to denote the set of rationals.

1. Examine the integrability of the following functions.

   (a) A continuous function $f : Q \to \mathbb{R}$.

   (b) An increasing function $f : [a, b] \to \mathbb{R}$.

   (c) A bounded function $f : Q \to \mathbb{R}$ such that $\{x \in Q : f(x) \neq 0\}$ is a closed set of measure 0.

   (d) $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

   $$f(x, y) = \begin{cases} 0, & \text{if } y \neq x, \text{ and} \\ 1, & \text{if } y = x. \end{cases}$$

   (e) $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

   $$f(x, y) = \begin{cases} 0, & \text{if } x \in [0, 1/2), \text{ and} \\ 1, & \text{if } x \in [1/2, 1]. \end{cases}$$

   (f) $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

   $$f(x, y) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q},\ y \in \mathbb{R} \setminus \mathbb{Q}, \text{ and} \\ 1/q, & \text{if } x \in \mathbb{Q},\ y = p/q \text{ in lowest terms.} \end{cases}$$

2. Examine whether the following subsets of $\mathbb{R}^n$ are of measure 0.

   (a) The set ∂U, where $U \subset [0,1]$ that is a union of open intervals $I_j = (a_j, b_j)$ such that each rational in $[0,1] \cap \mathbb{Q}$ is contained in I_j for some j.

   (b) For a continuous $f : [a,b] \to \mathbb{R}$, the set

   $$G_f = \{(x, f(x)) : x \in [a,b]\}.$$

   (c) The set $[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$.

   (d) The set $\mathbb{R}^{n-1} \times \{0\}$.

   (e) Any nontrivial open subset.

   (f) For a set $A \subset \mathbb{R}^n$ of measure 0, the associated sets $\bar{A}$, A°, and ∂A.

3. In each of the following problems, assume that $f : Q \to \mathbb{R}$ is integrable.

   (a) Show that $|f|$ is integrable and $|\int_Q f| \leq \int_Q |f|$.

   (b) For a $g : Q \to \mathbb{R}$, if $g = f$, except at finitely many points, then g is also integrable and $\int_Q f = \int_Q g$.

   (c) For an integrable $g : Q \to \mathbb{R}$ such that $f \leq g$, show that $\int_Q f \leq \int_Q g$

4. For a set $C \in \mathbb{R}^n$, the *characteristic function* on C is defined as

   $$\chi_C(x) := \begin{cases} 1, & \text{if } x \in C, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Reading assignment: Read Theorems 3-5, 3-6, and 3-9 from *Calculus on manifolds* by M. Spivak.

   (b) Give an example of a bounded set C of measure 0 such that $\int_Q \chi_C$ does not exist.

   (c) If C is a bounded set of measure 0 and $\int_Q \chi_C$ exists, then $\int_Q \chi_C = 0$.

   (d) Show show that $f = \chi_U$ (U as in problem 2(a)) except on a set of measure 0, then f is not integrable in $[0,1]$.

5. If $f : [a,b] \to \mathbb{R}$ is integrable and non-negative, show that, then show that $A_f = [a,b] \times [0, f(x)]$ is Jordan-measurable with area $\int_{[a,b]} f$.

6. If $f : [a,b] \times [a,b] \to \mathbb{R}$ is continuous, then show that

   $$\int_{y \in [a,b]} \int_{x \in [a,y]} f(x,y) = \int_{x \in [a,b]} \int_{y \in [x,b]} f(x,y).$$

7. Let $A \subset \mathbb{R}^2$ be open and let $f : A \to \mathbb{R}$ be of class C^2. For a rectangle $Q \subset A$, use Fubini theorem to show that

   $$\int_Q D_2 D_1 f = \int_Q D_1 D_2 f.$$

8. Use Fubini Theorem to derive an expression for the volume of a set in $\mathbb{R}^3$ obtained by revolving a Jordan-measurable set in the yz-plane about the z-axis.

9. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and let $D_2 f$ be continuous. If $F(y) = \int_{x\in[a,b]} f(x,y)$, then show that $F'(y) = \int_{x\in[a,b]} D_2 f(x,y)$.

10. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 such that $D_1 g_2 = D_2 g_1$. If

$$f(x,y) = \int_{t\in[0,x]} g_1(t,0) + \int_{t\in[0,y]} g_2(x,t),$$

then show that $D_1 f(x,y) = g_1(x,y)$.

11. Reading assignment: Read the proof of Theorem 3-14 from *Calculus on manifolds* by M. Spivak.